Point interactions of the dipole type defined through a three-parametric power regularization

2010 J. Phys. A: Math. Theor. 43 105302

(http://iopscience.iop.org/1751-8121/43/10/105302)

J. Phys. A: Math. Theor. **43** (2010) 105302 (21pp) doi:10.1088/1751-8113/43/10/105302

Point interactions of the dipole type defined through a three-parametric power regularization

A V Zolotaryuk

Bogolyubov Institute for Theoretical Physics, National Academy of Sciences of Ukraine, 03680 Kyiv, Ukraine

E-mail: azolo@bitp.kiev.ua

Received 10 December 2009, in final form 26 January 2010
Published 18 February 2010
Online at stacks.iop.org/JPhysA/43/105302

Abstract

A family of point interactions of the dipole type is studied in one dimension using a regularization by rectangles in the form of a barrier and a well separated by a finite distance. The rectangles and the distance are parametrized by a squeezing parameter $\varepsilon \to 0$ with three powers μ, ν and τ describing the squeezing rates for the barrier, the well and the distance, respectively. This parametrization allows us to construct a whole family of point potentials of the dipole type including some other point interactions, such as e.g. δ-potentials. Varying the power τ, it is possible to obtain in the zero-range limit the following two cases: (i) the limiting δ'-potential is opaque (the conventional result obtained earlier by some authors) or (ii) this potential admits a resonant tunneling (the opposite result obtained recently by other authors). The structure of resonances (if any) also depends on a regularizing sequence. The sets of the $\{\mu, \nu, \tau\}$-space where a non-zero (resonant or non-resonant) transmission occurs are found. For all these cases in the zero-range limit the transfer matrix is shown to be of the form $\Lambda = \begin{pmatrix} \chi & 0 \\ g & \chi^{-1} \end{pmatrix}$ with real parameters χ and g depending on a regularizing sequence. Those cases when $\chi \neq 1$ and $g \neq 0$ mean that the corresponding δ'-potential is accompanied by an effective δ-potential.

PACS numbers: 03.65.−w, 03.65.Db, 03.65.Ge

1. Introduction

Point and contact interactions are widely used in various areas of quantum physics, acoustics and optics (see [1–3] and references therein). In many cases these interactions are modeled by the one-dimensional Schrödinger equation:

$$-\psi''(x) + V(x)\psi(x) = E\psi(x) \tag{1}$$

with a zero-range singular potential $V(x)$ admitting exact solutions. Here the prime stands for the differentiation with respect to the spatial coordinate x and $\psi(x)$ is the wavefunction for a particle of mass m and energy E (we use units in which $\hbar^2/2m = 1$). Applications of these models to condensed matter physics (see, e.g., [4–7]), including very recent studies on curved quantum waveguides (see [8, 9] and the references therein), are of particular interest nowadays, mainly because of the rapid progress in fabricating nanoscale quantum devices.

This paper aims to study the family of point potentials $V(x)$ which are regularized by finite-range sequences of the dipole type $V_\varepsilon(x) \doteq \lambda\Delta'_\varepsilon(x)$ with λ being a coupling constant and ε a squeezing parameter. In simple terms, a regularized potential $V_\varepsilon(x)$ is supposed to consist of a barrier and a well, the height and the depth of which tend to infinity while their width is going to zero. In particular, the derivative of Dirac's delta function $\delta(x)$, i.e. the potential

$$V(x) = \lambda\delta'(x), \qquad \delta'(x) \doteq \mathrm{d}\delta(x)/\mathrm{d}x, \tag{2}$$

belongs to this family. In this case any regularizing sequence $\Delta'_\varepsilon(x)$ must satisfy the same integral properties as the $\delta'(x)$ function itself, i.e.

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} \Delta'_\varepsilon(x)\,\mathrm{d}x = 0 \qquad \text{and} \qquad \lim_{\varepsilon\to 0}\int_{\mathbb{R}} x\Delta'_\varepsilon(x)\,\mathrm{d}x = -1. \tag{3}$$

Until recently there was an opinion that potential (2) does not allow any transmission reflecting an incident quantum particle at all energies [10–12] resulting in separated particle states on the left ($\mathbb{R}^-$) and the right ($\mathbb{R}^+$) half-lines. However, recently in a series of papers [13–18] the existence of discrete resonance sets was established in the λ-space at which the transmission across barrier (2) becomes non-zero leading to the existence of non-separated states. More precisely, if the distribution $\delta'(x)$ is appropriately regularized by a sequence of finite-range functions $\Delta'_\varepsilon(x)$ with a squeezing parameter ε through the limit $\Delta'_\varepsilon(x) \to \delta'(x)$ in the sense of distributions, then in the zero-range limit ($\varepsilon \to 0$) equation (1) with potential (2) admits a countable set of resonances $\{\lambda_n\}_{n=1}^\infty$ with a partial transparency. Moreover, as shown for some particular cases of regularizing sequences $\Delta'_\varepsilon(x)$ [14, 15] and proved rigorously in a general case [17, 18], the structure of resonance sets depends on the $\Delta'_\varepsilon(x)$ profile. Outside this set, potential (2) is opaque acting as a perfect wall. The set of resonant potentials of another type for which the average over $\mathbb{R}$ is non-zero, contrary to the first limit in (3), have been obtained recently in [8, 9].

The reason why in some cases the $\delta'(x)$ barrier was proved to be opaque and in other cases it was observed as a resonantly transparent system can be explained by the following example. Let us construct the regularizing sequence consisting of a rectangular barrier and a rectangular well both with width l which are separated by some non-zero distance ρ. More precisely, we define the profile of this sequence as

$$\Delta'_{l\rho}(x) = \frac{1}{l(l+\rho)}\left[u\left(-\frac{x}{l}\right) - u\left(\frac{x-\rho}{l}\right)\right], \tag{4}$$

where $u(\xi) = 1$ if $\xi \in (0, 1)$ and $u(\xi) = 0$ otherwise. Here l and ρ serve as two independent squeezing parameters. Particularly, both the repeated distributional limits of function (4) give the $\delta'(x)$ function, i.e.

$$\lim_{\rho\to 0}\lim_{l\to 0}\Delta'_{l\rho}(x) = \lim_{l\to 0}\lim_{\rho\to 0}\Delta'_{l\rho}(x) = \delta'(x). \tag{5}$$

However, the study of scattering properties using both these ways of regularization in equation (1) with potential (2) gives different results. The first limit which can also be represented as

$$\delta'(x) = \lim_{\varepsilon\to 0}\Delta'_\varepsilon(x) = \lim_{\varepsilon\to 0}\frac{\delta(x+\varepsilon) - \delta(x-\varepsilon)}{2\varepsilon} \tag{6}$$

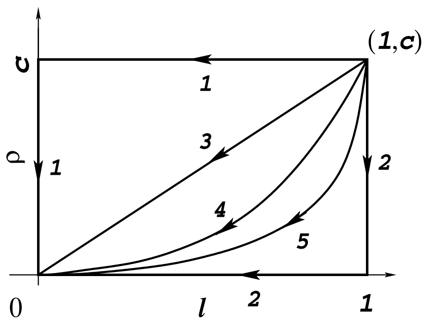

Figure 1. Diagram of different ways of regularization of the $\delta'(x)$ function shown by five paths $1, \ldots, 5$ starting from the same rectangular profile given by the parameter values $l = 1$ and $\rho = c$.

results in the full reflection of an incident particle from the $\delta'(x)$ barrier as calculated explicitly in [11] while the second limit (5) leads to the existence of a discrete resonance set $\{\lambda_n\}_{n=1}^{\infty}$ in the λ-space where the transmission is non-zero [13]. Surprisingly, both the results are correct and the solution of this riddle is that each of these results depends on the structure of a regularizing sequence. Schematically, both repeated limits (5) can be illustrated by two paths 1 and 2 shown in figure 1 starting from the same initial point, say $(l, \rho) = (1, c)$ with some $c > 0$. Along path 1 the $\delta'(x)$ barrier occurs to be fully reflecting, while following path 2 one obtains a resonant tunneling. Clearly, the $\delta'(x)$ function can be obtained from this initial point by many other ways, such as path 3 ($\rho = cl$), path 4 ($\rho = cl^2$) or path 5 ($\rho = cl^{\tau}$ with $\tau > 2$) and so on. One can expect that any sufficiently fast squeezing of distance ρ compared to squeezing width l will result in the existence of a resonance set and in the opposite case the $\delta'(x)$ barrier will be opaque for all $\lambda \neq 0$. The goal of the present paper is to construct a whole family of regularizing sequences, such as paths 3, 4 or 5 shown in figure 1, and to study the existence and structure of resonance sets in the zero-range limit. Note that the case of resonant tunneling through the second distributional limit (5) and illustrated by path 2 in figure 1 can be represented as a particular example of a general regularizing profile $\Delta'_\varepsilon(x)$ in the form

$$\delta'(x) = \lim_{\varepsilon\to 0} \Delta'_\varepsilon(x) = \lim_{\varepsilon\to 0} \varepsilon^{-2} v(x/\varepsilon) \tag{7}$$

with any function $v(\xi)$ satisfying the 'dipole' conditions $\int_{\mathbb{R}} v(\xi)\,\mathrm{d}\xi = 0$ and $\int_{\mathbb{R}} \xi v(\xi)\,\mathrm{d}\xi = -1$ [17, 18]. However, the existence of resonance sets for the particular case with $l = \varepsilon$ and $\rho = 0$ where $v(\xi) = u(-\xi) - u(\xi)$ [13] and for a general form of $v(\xi)$ [17] conflicts with the widely cited Šeba's result (see theorem 4 of [10]). Therefore, very recently Golovaty and Hryniv [18] have re-examined the proof of Šeba's theorem and, as a result, they have found that the $\delta'(x)$ barrier is not necessarily opaque, having clarified thus the situation concerning these controversial results. Another particular case of regularization (4) which reduces to the distributional limit of type (7) is the situation when $l = \varepsilon$ and $\rho = c\varepsilon$ [16] with any positive constant c. Here $v(\xi) = (1 + c)^{-1}[u(-\xi) - u(\xi - c)]$ and this case is illustrated in figure 1 by path 3.

The regularization procedure based on the potentials $\Delta'_\varepsilon(x)$ consisting of rectangles [13, 14, 16] also allows us to follow the way how the cancellation of divergences emerging from the kinetic energy operator $-\mathrm{d}^2/\mathrm{d}x^2$ and the product $\lambda\delta'(x)\psi(x)$ is accomplished explicitly in the $\varepsilon \to 0$ limit resulting in the total Hamiltonian defined on the wavefunctions $\psi(x)$

discontinuous with its derivatives at $x = 0$. Therefore, the limiting total Hamiltonian is no longer the sum of the kinetic and potential terms. More precisely, as found in [14], the boundary conditions (with accuracy to a constant) at $x = \pm 0$ in the zero-range limit are of the form

$$\psi(-0) = \chi^{-1}, \qquad \psi(+0) = 1, \qquad \psi'(-0) = \mathrm{i}k\chi - g, \qquad \psi'(+0) = \mathrm{i}k, \tag{8}$$

where the functions $\chi = \chi(\lambda) \neq 1$ and $g = g(\lambda)$ take finite values only at the resonances $\lambda = \lambda_n, n \in \mathbb{N}$. Equations (8) can be rewritten through the transfer matrix $\Lambda = \Lambda(\lambda)$ which connects the boundary conditions at $x = -0$ and $x = +0$:

$$\begin{pmatrix} \psi(+0) \\ \psi'(+0) \end{pmatrix} = \Lambda \begin{pmatrix} \psi(-0) \\ \psi'(-0) \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \chi & 0 \\ g & \chi^{-1} \end{pmatrix}. \tag{9}$$

Due to the dependence of the resonance sets $\{\lambda_n\}_{n=1}^{\infty}$ on the regularizing families $\Delta'_\varepsilon(x)$, one can claim on the existence of a mapping $\Delta'_\varepsilon(x) \longmapsto \{\Lambda_n\}_{n=1}^{\infty}$ where $\Lambda_n \doteq \Lambda(\lambda_n)$ in the resonant case.

Using equation (9) and the standard representation

$$\psi(x) = \begin{cases} \mathrm{e}^{\mathrm{i}kx} + R\,\mathrm{e}^{-\mathrm{i}kx} & \text{for} \quad x < 0, \\ T\,\mathrm{e}^{\mathrm{i}kx} & \text{for} \quad x > 0, \end{cases} \tag{10}$$

for the reflection (R) and transmission (T) coefficients, one can write these coefficients in terms of χ and g as follows:

$$R = \frac{\chi^{-1} - \chi - \mathrm{i}g/k}{\chi^{-1} + \chi + \mathrm{i}g/k} \qquad \text{and} \qquad T = \frac{2}{\chi^{-1} + \chi + \mathrm{i}g/k}. \tag{11}$$

As calculated in [14] and it will also be shown below for other cases that $g = \infty$ outside the resonance values and therefore here we have $R = -1$ and $T = 0$, i.e. the complete reflection. The case with $\chi = 1$ and $g \neq 0$ corresponds to the pure δ-potential, while in the opposite case with $\chi \neq 1$ and $g = 0$ we have the pure δ'-potential. There exists the possibility [14] that $g \neq 0$ (and also $\chi \neq 1$) for some families $\Delta'_\varepsilon(x)$ in the resonant case. For this case one can claim that the point interaction regularized by a sequence of finite-range dipole-like potentials is accompanied by a δ-interaction.

It is worth mentioning here that besides the point interactions described by matrix (9) where $\chi \neq 1$, there exists another type called 'the δ'-interaction' which has been proposed by Albeverio *et al* in [2, 3] (for further discussion on this point interaction see also [10, 12, 17, 18]). Contrary to equation (9), the matrix Λ for the δ'-interaction with the strength interaction constant β is given by the elements $\Lambda_{11} = \Lambda_{22} = 1$ and $\Lambda_{12} = \beta$, so that at $x = 0$ the wavefunctions $\psi(x)$ are discontinuous while their derivatives are continuous: $\psi'(-0) = \psi'(+0) \doteq \psi'(0)$ and $\psi(+0) - \psi(-0) = \beta\psi'(0)$. Therefore, in order to avoid any confusion throughout the paper we use only 'the δ'-potential' term for potential (2) having kept 'the δ'-interaction' notation for the point interaction introduced in [2, 3].

Clearly, boundary conditions (9) are invariant under the transformation $\psi(\pm 0) \to \chi\psi(\mp 0)$ and $\psi'(\pm 0) \to \chi^{-1}\psi'(\mp 0)$. They form a subfamily of the whole family of non-separated connection matrices [19]:

$$\Lambda = \mathrm{e}^{\mathrm{i}\vartheta} \begin{pmatrix} \lambda_{11}\lambda_{12} \\ \lambda_{21}\lambda_{22} \end{pmatrix} \tag{12}$$

with the real parameters $\vartheta \in [0, \pi)$ and $\lambda_{ij} \in \mathbb{R}, i, j = 1, 2$, fulfilling the equation $\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21} = 1$.

In the present paper we define a more general class of point interactions than the $\delta'(x)$ potential given by (2), generalizing the approach of Šeba [10] from one to three dimensions.

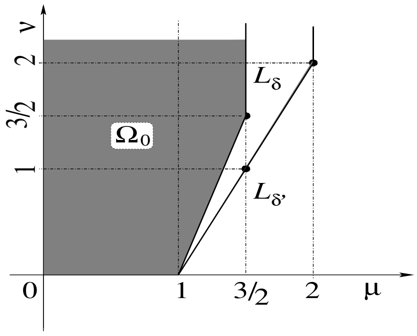

Figure 2. Regions of existence and non-existence of non-zero transparency on the $\{\mu, \nu\}$-plane. Non-trivial point interactions are located on the lines $L_\delta \doteq \Omega_1 \cup \Omega_2 \cup \Omega_3$ and $L_{\delta'} \doteq \Omega_4 \cup \Omega_5 \cup \Omega_6$.

To this end, we parametrize the regularizing barrier–well rectangles through powers. For the case of two dimensions this approach has already been developed in [14] where the barrier of height h and width l and the adjacent well of depth d and width r have been parametrized as

$$l = \varepsilon, \qquad h = a\varepsilon^{-\mu}, \qquad d = b\varepsilon^{-\nu}, \qquad r = \eta\varepsilon^{1-\mu+\nu}, \qquad \eta \doteq a/b, \tag{13}$$

with any positive constants a, b, μ and ν. In the zero-range limit $\varepsilon \to 0$ the region

$$\Omega_0 \doteq \{1 < \mu < 3/2, \nu > 3(\mu - 1)\} \tag{14}$$

(see figure 2) is fully transparent ($T = 1$), while at its boundary $L_\delta = \Omega_1 \cup \Omega_2 \cup \Omega_3$ where

$$\begin{aligned} &\Omega_1 \doteq \{1 < \mu < 3/2,\ \nu = 3(\mu - 1)\}, \\ &\Omega_2 \doteq \{\mu = 3/2, \nu > 3/2\}, \qquad \Omega_3 \doteq \{\mu = \nu = 3/2\} \end{aligned} \tag{15}$$

the system behaves effectively as a δ-interaction, i.e. $\chi = 1$ and $g \neq 0$ in the connection matrix Λ defined by equation (9). Note that the system allows a full transmission in the region $\{1 < \mu < 3/2,\ \nu > 0\}$ with stronger singularity than the δ-potential. This transmission with $\Lambda = I$ where I is the matrix unit occurs due to the presence of a well resulting in the cancellation of singularities as $\varepsilon \to 0$. Outside the region $\Omega_0 \cup L_\delta$ potential (2) is opaque ($T = 0$), except for the sets

$$\begin{aligned} &\Omega_4 \doteq \{1 < \mu < 3/2, \nu = 2(\mu - 1)\}, \\ &\Omega_5 \doteq \{\mu = 3/2, \nu = 1\}, \qquad \Omega_6 \doteq \{\mu = \nu = 2\}, \end{aligned} \tag{16}$$

where the resonant tunneling mentioned above occurs, but this result is obtained if we follow path 2 in figure 1. The sets are subsets of the δ'-line $L_{\delta'} \doteq \Omega_6 \cup \Omega_7 \cup \Omega_8$ (see figure 2) where

$$\Omega_7 \doteq \{1 < \mu \leqslant 2, \nu = 2(\mu - 1)\} \qquad \Omega_8 \doteq \{\mu = 2, 2 < \nu < \infty\}. \tag{17}$$

Thus, using the $\{\mu, \nu\}$-parametrization given by equations (13), one can describe the family of all δ-interactions generalizing Šeba's result [10] for the point $\mu = \nu = 3/2$ to the whole line L_δ. The resonant tunneling occurs on the line $L_{\delta'}$ being the set of all $\varepsilon \to 0$ limits for the $\delta'(x)$ function.

In this paper we construct the regularizing families adding a third parameter τ which describes the squeezing rate of the barrier–well distance ρ to the powers μ and ν in equations (13). This parameter measures how rapidly a barrier and a well in the $\Delta'_\varepsilon(x)$ profile approach each other. To illustrate our study with calculations, we choose the regularizing sequence $\Delta'_\varepsilon(x)$ to consist of rectangles. In this case we are able to control the cancellation of divergences emerging from the kinetic energy term and the singular potential. When this cancellation occurs, the connection matrix Λ takes the form of (9). It is important to note that the cancellation of divergences can be accomplished in different ways leading to different families of matrices Λ.

2. A finite-range solution and its parametrization

Thus, we consider the starting situation when a barrier of height h and width l and a well of depth d and width r are supposed to be at a distance ρ. Assume all these parameters to depend on a parameter $\varepsilon > 0$ in such a way that the parameters l, r and ρ tend to zero while the height h and the depth d go to infinity as $\varepsilon \to 0$ forming a zero-range potential $V(x)$ in equation (1). Consequently, the regularizing sequence is defined as follows:

$$\Delta'_\varepsilon(x) \doteq \begin{cases} 0 & \text{for} \quad x \in (-\infty, -l), (0, \rho), (\rho + r, \infty), \\ h & \text{for} \quad x \in (-l, 0), \\ -d & \text{for} \quad x \in (\rho, \rho + r), \end{cases} \tag{18}$$

where the ε-dependence will be specified below.

By lengthy but straightforward calculations one can represent a finite-range solution of equation (1) with the regularized potential $V_\varepsilon(x) = \lambda \Delta'_\varepsilon(x)$, where $\Delta'_\varepsilon(x)$ is defined by (18), through the transfer matrix Λ connecting the boundary conditions at $x = -l$ and $x = \rho + r$:

$$\begin{pmatrix} \psi(\rho + r) \\ \psi'(\rho + r) \end{pmatrix} = \Lambda \begin{pmatrix} \psi(-l) \\ \psi'(-l) \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \Lambda_{11} \Lambda_{12} \\ \Lambda_{21} \Lambda_{22} \end{pmatrix}. \tag{19}$$

The matrix elements Λ_{ij}, $i, j = 1, 2$, are given by

$$\begin{aligned} \Lambda_{11} = {} & \left[\cosh(pl)\cos(qr) + \frac{p}{q}\sinh(pl)\sin(qr)\right]\cos(k\rho) \\ & + \left[\frac{p}{k}\sinh(pl)\cos(qr) - \frac{k}{q}\cosh(pl)\sin(qr)\right]\sin(k\rho), \\ \Lambda_{12} = {} & \left[\frac{1}{p}\sinh(pl)\cos(qr) + \frac{1}{q}\cosh(pl)\sin(qr)\right]\cos(k\rho) \\ & + \left[\frac{1}{k}\cosh(pl)\cos(qr) - \frac{k}{pq}\sinh(pl)\sin(qr)\right]\sin(k\rho), \\ \Lambda_{21} = {} & [p\sinh(pl)\cos(qr) - q\cosh(pl)\sin(qr)]\cos(k\rho) \\ & - \left[k\cosh(pl)\cos(qr) + \frac{pq}{k}\sinh(pl)\sin(qr)\right]\sin(k\rho), \\ \Lambda_{22} = {} & \left[\cosh(pl)\cos(qr) - \frac{q}{p}\sinh(pl)\sin(qr)\right]\cos(k\rho) \\ & - \left[\frac{k}{p}\sinh(pl)\cos(qr) + \frac{q}{k}\cosh(pl)\sin(qr)\right]\sin(k\rho) \end{aligned} \tag{20}$$

with $k \doteq \sqrt{E}$, $p \doteq \sqrt{\lambda h - E}$ and $q \doteq \sqrt{\lambda d + E}$.

Now we specify the power parametrization of h, d, l, r and ρ using the single squeezing parameter ε as follows:

$$l = \varepsilon, \qquad h = a\varepsilon^{-\mu}, \quad r = r_0\varepsilon^{\gamma}, \qquad d = b\varepsilon^{-\nu}, \qquad \rho = c\varepsilon^{\tau} \tag{21}$$

with positive coefficients a, b, c, r_0 and positive powers μ, ν, τ, γ. The parameter γ will be determined below in terms of μ, ν and τ for each particular case. The case $c = 0$ is also available and it means that in the regularization procedure the second repeated limit (5) is assumed. With the notation $\sigma \doteq \sqrt{a\lambda}$, for any positive μ and ν we expand pl and qr and obtain

$$\begin{aligned} pl &= \sigma \varepsilon^{1-\mu/2} \left[1 - \frac{k^2}{2\sigma^2} \varepsilon^{\mu} + \mathcal{O}(\varepsilon^{2\mu}) \right], \\ qr &= \frac{\rho\sigma}{\sqrt{\eta}} \varepsilon^{\gamma-\nu/2} \left[1 + \frac{\eta k^2}{2\sigma^2} \varepsilon^{\nu} + \mathcal{O}(\varepsilon^{2\nu}) \right]. \end{aligned} \tag{22}$$

Our goal is to find the conditions under which the zero-range limit ($\varepsilon \to 0$) of all the elements $\Lambda_{ij} = \Lambda_{ij}(\varepsilon)$ given by equations (20) is finite. Because of the form of equations (20), it is convenient to consider the $\varepsilon \to 0$ limit separately for the following four cases: (i) $pl \to 0$ and $qr \to 0$, (ii) $pl \to 0$ but qr tends to a non-zero finite constant, (iii) both pl and qr tend to non-zero finite constants and (iv) pl goes to a non-zero finite constant while $qr \to 0$. In this way, one can split the $\{\mu, \nu, \tau\}$-space into several regions and analyze in each of these regions the asymptotic behavior of the transfer matrix Λ and the scattering amplitudes (reflection and transmission coefficients).

The matrix element Λ_{21} appears to be the most singular term in the region $\mu > 1$ as $\varepsilon \to 0$. It can be well defined only if an appropriate cancellation of singularities occurs. Therefore, the most direct way in the analysis is to start with the zero-range limit of Λ_{21}. To this end we expand this element in powers of ε and re-arrange this expansion as

$$\Lambda_{21} = \Lambda_{21}^{(0)} + \Lambda_{21}^{(1)} + \Lambda_{21}^{(2)} + \cdots, \tag{23}$$

where the group of terms $\Lambda_{21}^{(0)}$ contains the most singular expressions which under appropriate constraints cancel out as $\varepsilon \to 0$. Under these constraints in the form of equations a non-zero transmission across the limiting zero-range potential $V(x)$ occurs. The next group $\Lambda_{21}^{(1)}$ contains less singular terms and using here the equations for a non-zero transparency, one can find the regions in the $\{\mu, \nu, \tau\}$-space where $\lim_{\varepsilon\to 0} \Lambda_{21}^{(1)} \doteq g$ is finite. Under the conditions of a non-zero transmission the next terms $\Lambda_{21}^{(2)}$ and so on vanish in the $\varepsilon \to 0$ limit. Finally, using these conditions, one can calculate the other limits which appear to be finite, namely $\lim_{\varepsilon\to 0} \Lambda_{12} = 0$, $\lim_{\varepsilon\to 0} \Lambda_{11} \doteq \chi$ and $\lim_{\varepsilon\to 0} \Lambda_{22} = \chi^{-1}$. Thus, as will be shown below, including the third parameter τ leads to the same form of connection matrix Λ as in (9). The set of available values of the coupling constant λ at which a non-zero transmission occurs can be either continuous or discrete.

3. Point interactions with a non-zero and non-resonant transparency: case (i)

In this section we consider case (i) when both $pl \to 0$ and $qr \to 0$. The $pl \to 0$ limit implies the inequality $\mu < 2$ and since the singular behavior occurs only for $\mu > 1$, the region of interest is the interval $1 < \mu < 2$. The other limit $qr \to 0$ implies the inequality $\gamma > \nu/2$. We expand Λ_{21} in the form

$$\begin{aligned} \Lambda_{21} = p^2 l \left(1 + \frac{p^2l^2}{6}\right)\left(1 - \frac{q^2r^2}{2}\right) - q^2 r \left(1 + \frac{p^2l^2}{2}\right)\left(1 - \frac{q^2r^2}{6}\right) \\ - p^2 l \rho \left(1 + \frac{p^2l^2}{6}\right) q^2 r \left(1 - \frac{q^2r^2}{6}\right) + \cdots. \end{aligned} \tag{24}$$

In this expansion the most singular part $\Lambda_{21}^{(0)}$ can be arranged in two ways as follows.

3.1. An effective δ-interaction: subcase (ia)

First we consider the situation when the most singular part consists of the following two terms:

$$\Lambda_{21}^{(0)} = p^2 l - q^2 r = \sigma^2 \varepsilon^{1-\mu} - \frac{r_0 \sigma^2}{\eta} \varepsilon^{\gamma-\nu} + \cdots. \tag{25}$$

These terms cancel out in the $\varepsilon \to 0$ limit, resulting in the limit $\Lambda_{21}^{(0)} \to 0$ if $\gamma = 1 - \mu + \nu > 0$ and $r_0 = \eta$. Since $\gamma > \nu/2$, we have $1 - \mu + \nu > \nu/2$ leading to the inequality $\nu > 2(\mu - 1)$.

Using expansion (25), one can write the next part $\Lambda_{21}^{(1)}$ in expansion (24) in the form

$$\begin{aligned}\Lambda_{21}^{(1)} &= \frac{p^2 l}{2}\left(\frac{p^2 l^2}{3} - q^2 r^2\right) - \frac{q^2 r}{2}\left(p^2 l^2 - \frac{q^2 r^2}{3}\right) - p^2 l q^2 \rho r \\ &= -\frac{p^4 l^2}{3}(l + 3\rho + r) + \cdots \\ &= -\frac{\sigma^4}{3}(\varepsilon^{3-2\mu} + \eta \varepsilon^{3-3\mu+\nu} + 3c\varepsilon^{2-2\mu+\tau}) + \cdots.\end{aligned} \tag{26}$$

As follows from these asymptotics, the limiting matrix element Λ_{21} will be finite if the inequalities

$$3 - 2\mu \geqslant 0, \qquad 3 - 3\mu + \nu \geqslant 0, \qquad \tau \geqslant 2(\mu - 1) \tag{27}$$

hold simultaneously. These inequalities define the regions in the $\{\mu, \nu, \tau\}$-space where a non-zero transparency takes place as $\varepsilon \to 0$. These regions will be found below on the basis of inequalities (27).

As regards the other elements of the matrix Λ, due to equations (21) and inequalities (27), we obtain the zero limit of Λ_{12} and finite limits for Λ_{11} and Λ_{22}, namely $\Lambda_{11} \to 1$ and $\Lambda_{22} \to 1$. Thus, the connection matrix Λ in the $\varepsilon \to 0$ limit has the form of (9) with $\chi = 1$ and the constant g which can be calculated explicitly using asymptotics (26) with inequalities (27). The regions in the $\{\mu, \nu\,\tau\}$-space where g is finite exist if $\tau \geqslant 2(\mu - 1)$ and, as a result, we define the following sets (see figure 3):

$$\begin{aligned}
M_0 &\doteq \{1 < \mu < 3/2, \nu > 3(\mu - 1), \tau > 2(\mu - 1)\},\\
M_1 &\doteq \{1 < \mu < 3/2, \nu = 3(\mu - 1), \tau > 2(\mu - 1)\},\\
M_2 &\doteq \{\mu = 3/2, \nu > 3/2, \tau > 1\},\\
M_3 &\doteq \{\mu = \nu = 3/2, \tau > 1\},\\
N_0 &\doteq \{1 < \mu < 3/2, \nu > 3(\mu - 1), \tau = 2(\mu - 1)\},\\
N_1 &\doteq \{1 < \mu < 3/2, \nu = 3(\mu - 1), \tau = 2(\mu - 1)\},\\
N_2 &\doteq \{\mu = 3/2, \nu > 3/2, \tau = 1\},\\
N_3 &\doteq \{\mu = \nu = 3/2, \tau = 1\}.
\end{aligned} \tag{28}$$

Then the coupling constant g is given by the following values:

$$g = -\frac{a^2 \lambda^2}{3} \begin{cases} 0 & \text{for} \quad M_0 \text{ and } \Omega_0 \\ \eta & \text{for} \quad M_1 \text{ and } \Omega_1 \\ 1 & \text{for} \quad M_2 \text{ and } \Omega_2 \\ 1 + \eta & \text{for} \quad M_3 \text{ and } \Omega_3 \\ 3c & \text{for} \quad N_0 \\ \eta + 3c & \text{for} \quad N_1 \\ 1 + 3c & \text{for} \quad N_2 \\ 1 + \eta + 3c & \text{for} \quad N_3. \end{cases} \tag{29}$$

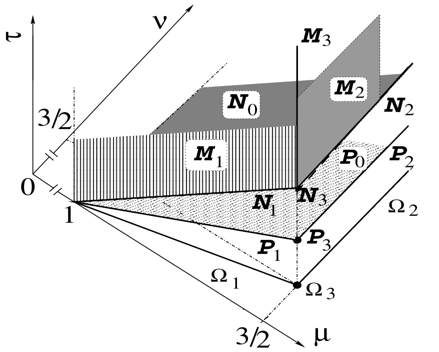

Figure 3. The regions of existence of non-zero transparency in the $\{\mu, \nu, \tau\}$-space including the particular case of two dimensions ($\tau \equiv 0$). In the $\tau \geqslant 2(\mu - 1)$ half-space planes M_1, M_2, N_0, lines M_3, N_1, N_2 and point N_3 form the trihedral surface S_δ where the system behaves as a δ-potential with the effective coupling constant g given by equations (29). In the interior of this trihedral M_0, not shown in figure, the system is completely transparent. The similar δ-interaction behavior occurs in the plane $\tau \equiv 0$ on lines Ω_1, Ω_2 and point Ω_3 forming the boundary L_δ of fully transparent region Ω_0. Plane P_0 and its boundary consisting of lines P_1, P_2 and point P_3 (all these sets lie in the $\tau = \mu - 1$ plane) correspond to δ'-potentials with the possible addition of an effective δ-interaction as described by equations (34).

Instead of the closed line L_δ, now we have the closed two-dimensional trihedral surface $S_\delta \doteq M_1 \cup M_2 \cup M_3 \cup N_0 \cup N_1 \cup N_2 \cup N_3$.

As follows from equations (29), the sets Ω_0 and M_0 are regions of complete transparency. On the boundaries of these sets, the transparency is partial for all $\lambda \neq 0$ being the same as in the case of the δ-potential with the coupling constant g given by equations (29). Outside the sets $\Omega_0 \cup L_\delta$ and $M_0 \cup S_\delta$, the transparency is zero and the system behaves as a perfect wall, except for some sets with a non-zero transmission to be considered below.

3.2. The δ'-potentials with a non-zero and non-resonant transparency: subcase (ib)

Let us now include into the most singular expression $\Lambda_{21}^{(0)}$ the first term from the group with ρ in expansion (24). Then, instead of equation (25), we find

$$\begin{aligned}\Lambda_{21}^{(0)} &= p^2 l - q^2 r - p^2 l q^2 \rho r \\ &= \sigma^2 \varepsilon^{1-\mu} - \frac{r_0}{\eta}\sigma^2 \varepsilon^{\gamma-\nu} - \frac{c r_0}{\eta}\sigma^4 \varepsilon^{1-\mu+\gamma-\nu+\tau} + \cdots.\end{aligned} \tag{30}$$

First, we note that the powers of ε at all the three terms have to be equal and this results in the equations $1 - \mu = \gamma - \nu = 1 - \mu + \gamma - \nu + \tau$ from which we immediately obtain the equations $\gamma = 1 - \mu + \nu$ and $\tau = \mu - 1$. Next, all the three terms in (30) cancel out if in addition the equation $r_0 = \eta/(1 + ac\lambda) < \eta$ holds.

Now, instead of expansion (26), we have to analyze the zero-range limit for the expression

$$\Lambda_{21}^{(1)} = \frac{p^2 l}{2}\left(\frac{p^2 l^2}{3} - q^2 r^2\right) - \frac{q^2 r}{2}\left(p^2 l^2 - \frac{q^2 r^2}{3}\right) - \frac{p^2 l q^2 r}{6}(p^2 l^2 - q^2 r^2)\rho + \cdots. \tag{31}$$

Using asymptotics (22), expansion (31) can be transformed into

$$\Lambda_{21}^{(1)} = -\frac{\sigma^4}{3(1 + c\sigma^2)}\left(\varepsilon^{3-2\mu} + \frac{\eta}{1 + c\sigma^2}\varepsilon^{3-3\mu+\nu}\right) + \cdots. \tag{32}$$

Concerning the other matrix coefficients, we have $\lim_{\varepsilon\to 0}\Lambda_{12} = 0$ and the asymptotics

$$\begin{aligned}\Lambda_{11} &= 1 + c\sigma^2 + \frac{\eta\sigma^2}{1+c\sigma^2}\varepsilon^{2-2\mu+\nu} + \cdots,\\ \Lambda_{22} &= \frac{1}{1+c\sigma^2} - \frac{\sigma^2}{1+c\sigma^2}\varepsilon^{2-\mu} + \cdots.\end{aligned} \tag{33}$$

Thus, the zero-range limit of equations (32) and (33) results in the connection matrix Λ of the same form as in (9) with

$$\chi = 1 + ac\lambda \qquad \text{and} \qquad g = -\frac{a^2\lambda^2}{3(1+ac\lambda)}\begin{cases}0 & \text{for} \quad P_0\\ \eta(1+ac\lambda)^{-1} & \text{for} \quad P_1\\ 1 & \text{for} \quad P_2\\ 1+\eta(1+ac\lambda)^{-1} & \text{for} \quad P_3.\end{cases} \tag{34}$$

Here the P-sets shown in figure 3 are defined from the condition of finiteness of asymptotics (32):

$$\begin{aligned}P_0 &\doteq \{1 < \mu < 3/2, \nu > 3(\mu-1), \tau = \mu - 1\},\\ P_1 &\doteq \{1 < \mu < 3/2, \nu = 3(\mu-1), \tau = \mu - 1\},\\ P_2 &\doteq \{\mu = 3/2, \nu > 3/2, \tau = 1/2\},\\ P_3 &\doteq \{\mu = \nu = 3/2, \tau = 1/2\}.\end{aligned} \tag{35}$$

On the plane P_0 we have a pure δ'-potential whereas the boundary line $L_{\delta\delta'} \doteq P_1 \cup P_2 \cup P_3$ corresponds to the mixture of δ- and δ'-potentials.

Thus, depending on within which set in the $\{\mu, \nu, \tau\}$-space the $\varepsilon \to 0$ limit is considered, the effective coupling constant g takes one of the values given by equations (29) or (34). Three of these values, namely those calculated for the M-sets, have been obtained previously in [14] for the particular case $\rho = 0$. As expected they appear to be the same for sufficiently large τ, i.e. under the condition $\tau > 2(\mu - 1)$.

4. Discrete resonance sets with $pl \to 0$ and $qr \to$ const > 0: case (ii)

As follows from the first equation (22), for non-trivial interactions (with $\mu > 1$) the limit $pl \to 0$ leads to the inequality $\mu < 2$, so for case (ii) we have to accomplish the asymptotic analysis within the interval $1 < \mu < 2$. From the second equation (22) we find that the limit $qr \to \text{const} \neq 0$ imposes the equation $\gamma = \nu/2$. Under these conditions, we expand

$$\begin{aligned}\Lambda_{21} &= p^2 l\left(1 + \frac{p^2l^2}{6}\right)\left[\cos\left(\frac{r_0\sigma}{\sqrt{\eta}}\right) + \mathcal{O}(l^\nu)\right]\\ &\quad - \left(1 + \frac{p^2l^2}{2}\right)\frac{\sigma}{\sqrt{\eta}}l^{-\nu/2}\left[\sin\left(\frac{r_0\sigma}{\sqrt{\eta}}\right) + \mathcal{O}(l^\nu)\right]\\ &\quad - p^2 l\rho\left(1 + \frac{p^2l^2}{6}\right)\frac{\sigma}{\sqrt{\eta}}l^{-\nu/2}\left[\sin\left(\frac{r_0\sigma}{\sqrt{\eta}}\right) + \mathcal{O}(l^\nu)\right] + \cdots.\end{aligned} \tag{36}$$

Similarly as in section 3, we analyze this expansion in the two subcases as follows.

4.1. *Resonances in the $\tau \geqslant 2(\mu - 1)$ half-space: subcase (iia)*

Consider first the situation when in expansion (36) one can arrange the most singular part in the form

$$\Lambda_{21}^{(0)} = \sigma^2\varepsilon^{1-\mu}\cos\left(\frac{r_0\sigma}{\sqrt{\eta}}\right) - \frac{\sigma}{\sqrt{\eta}}\varepsilon^{-\nu/2}\sin\left(\frac{r_0\sigma}{\sqrt{\eta}}\right). \tag{37}$$

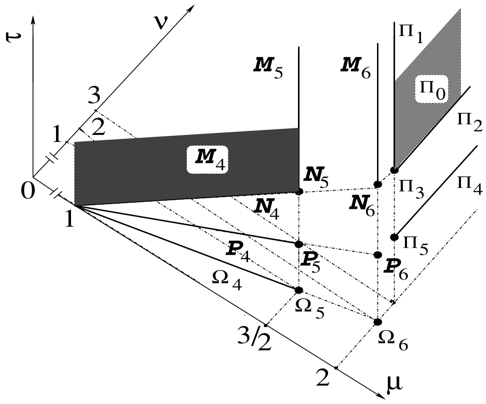

Figure 4. Regions of existence of non-zero transparency in the $\{\mu, \nu, \tau\}$-space including the particular case of two dimensions ($\tau \equiv 0$) for cases (ii), (iii) and (iv). In the $\tau \geqslant 2(\mu - 1)$ half-space and the $\tau \equiv 0$ plane (a-subcases): plane M_4 (line Ω_4), lines M_5 (point Ω_5) and N_4, point N_5; line M_6 (point Ω_6) and point N_6; plane Π_0, lines Π_1 and Π_2, point Π_3 are the sets for the resonances given by equations (38), (51) and (61), respectively. In the $\tau = \mu - 1$ plane (b-subcases): plane Π_0, lines Π_1 and Π_2, point Π_3; point P_6; line Π_4 and point Π_5 are the sets for the resonances given by equations (45), (57) and (67), respectively.

The cancellation of divergences in expansion (37) as $\varepsilon \to 0$ occurs if $1 - \mu = -\nu/2$, resulting in the relation $\nu = 2(\mu - 1)$ and the equation

$$\tan\left(\frac{r_0\sigma}{\sqrt{\eta}}\right) = \sqrt{\eta}\sigma. \tag{38}$$

The last equation admits a countable set of roots $\{\sigma_n\}_{n=1}^{\infty}$ (resonances), each of which depends on two parameters: $\sigma_n = \sigma_n(\eta, r_0)$. Therefore, in the λ-space each solution depends on three parameters: $\lambda_n = \lambda_n(a, b, r_0)$.

Using the equations $\nu = 2(\mu - 1)$ and (38), the next term $\Lambda_{21}^{(1)}$ in expansion (36) can be represented as

$$\Lambda_{21}^{(1)} = -\frac{\sigma_n^4}{3}\cos\left(\frac{c_2\sigma_n}{\sqrt{\eta}}\right)(l^{3-2\mu} + 3cl^{2-2\mu+\tau}) + \cdots. \tag{39}$$

Thus, the limit Λ_{21} is finite (zero or a non-zero constant) if in addition to the equation $\nu = 2(\mu - 1)$ the inequalities

$$1 < \mu \leqslant 3/2, \qquad \tau \geqslant 2(\mu - 1) \tag{40}$$

are satisfied. Concerning the other elements of the matrix Λ, under the equation $\nu = 2(\mu - 1)$ and inequalities (40), and using equation (38) for discrete resonances, we get the $\varepsilon \to 0$ limits: $\Lambda_{12} \to 0$, $\Lambda_{11} \to \chi$ and $\Lambda_{22} \to \chi^{-1}$ with

$$\chi = \cos\left(\frac{r_0\sigma_n}{\sqrt{\eta}}\right)^{-1} = \frac{\sqrt{\eta}\,\sigma_n}{\sin\left(r_0\sigma_n/\sqrt{\eta}\right)} = (-1)^n\sqrt{1 + a^2\lambda_n/b}. \tag{41}$$

Similar to case (i), the situation can be generalized to include the particular case with $c = 0$ ($\rho = 0$). In this case, one can put $\tau = 0$ and consider the sets Ω_4 and Ω_5 defined by equations (16) and shown in figure 4. For $c > 0$ in the $\tau \geqslant 2(\mu - 1)$ half-plane, one can define additionally the following regions:

$$\begin{aligned} M_4 &\doteq \{1 < \mu < 3/2,\, \nu = 2(\mu - 1),\, \tau > 2(\mu - 1)\}, \\ M_5 &\doteq \{\mu = 3/2,\, \nu = 1,\, \tau > 1\}, \\ N_4 &\doteq \{1 < \mu < 3/2,\, \nu = 2(\mu - 1),\, \tau = 2(\mu - 1)\}, \\ N_5 &\doteq \{\mu = 3/2,\, \nu = 1,\, \tau = 1\}. \end{aligned} \tag{42}$$

Then the matrix element $g \doteq \lim_{\varepsilon\to 0} \Lambda_{21}$ is given at the resonance set $\{\lambda_n\}_{n=1}^{\infty}$ by the equations

$$g = -\frac{a^2\lambda_n^2}{3}\cos(r_0\sqrt{b\lambda_n}) \begin{cases} 0 & \text{for} \quad M_4 \text{ and } \Omega_4 \\ 1 & \text{for} \quad M_5 \text{ and } \Omega_5 \\ 3c & \text{for} \quad N_4 \\ 1 + 3c & \text{for} \quad N_5. \end{cases} \tag{43}$$

Thus, in the zero-range limit we have obtained the matrix Λ in form (9) with χ and g given by equations (41) and (43), respectively.

4.2. Resonances in the $\tau = \mu - 1$ plane: subcase (iib)

Another possibility of the cancellation of divergences in the Λ_{21}-term can be provided if we add the first term in the expansion for the ρ-term in (36) to the group of terms $\Lambda_{21}^{(0)}$ given by equation (37), so now three terms are involved in the cancellation procedure. Then instead of (37), we have the expression

$$\Lambda_{21}^{(0)} = \sigma^2\varepsilon^{1-\mu}\cos\left(\frac{r_0\sigma}{\sqrt{\eta}}\right) - \frac{\sigma}{\sqrt{\eta}}\varepsilon^{-\nu/2}\sin\left(\frac{r_0\sigma}{\sqrt{\eta}}\right) - \frac{c\sigma^3}{\sqrt{\eta}}\varepsilon^{1-\mu-\nu/2+\tau}\sin\left(\frac{r_0\sigma}{\sqrt{\eta}}\right). \tag{44}$$

From the equality of powers of ε at all the three terms in this expression we obtain the two equalities $1 - \mu = -\nu/2 = \tau + 1 - \mu - \nu/2$ from which the equations $\nu = 2(\mu - 1)$ and $\tau = \mu - 1$ follow. In addition to these equations, similar to equation (38), the cancellation of divergences completely occurs for the resonance set $\{\sigma_n\}_{n=1}^{\infty}$ obeying the condition

$$\tan\left(\frac{r_0\sigma}{\sqrt{\eta}}\right) = \frac{\sqrt{\eta}\,\sigma}{1 + c\sigma^2}. \tag{45}$$

Note that the corresponding equation (38) for resonances with $\tau \geqslant 2(\mu - 1)$ can be obtained from equation (45) by putting $c = 0$.

Using equation for discrete resonances (45), the remaining part in expansion (36) can be written as

$$\Lambda_{21}^{(1)} = -\frac{\sigma_n^4}{3\left(1 + c\sigma_n^2\right)}\cos\left(\frac{r_0\sigma_n}{\sqrt{\eta}}\right)\varepsilon^{3-2\mu} + \cdots. \tag{46}$$

Similarly, we define the sets

$$\begin{aligned} P_4 &\doteq \{1 < \mu < 3/2,\, \nu = 2(\mu - 1),\, \tau = \mu - 1\}, \\ P_5 &\doteq \{\mu = 3/2,\, \nu = 1,\, \tau = 1/2\} \end{aligned} \tag{47}$$

shown in figure 4. The matrix element $g = g(\lambda_n)$ calculated at the resonance set $\{\lambda_n\}_{n=1}^{\infty}$ is

$$g = -\frac{a^2\lambda_n^2}{3(1 + ac\lambda_n)}\cos(r_0\sqrt{b\lambda_n}) \begin{cases} 0 & \text{for} \quad P_4 \\ 1 & \text{for} \quad P_5. \end{cases} \tag{48}$$

Similarly, concerning the other elements of the matrix $\Lambda = \Lambda(\lambda_n)$, under the conditions $\nu = 2(\mu - 1)$ and $\tau = \mu - 1$ and using equation (45), we obtain $\Lambda_{12} \to 0$, $\Lambda_{11} \to \chi$ and $\Lambda_{22} \to \chi^{-1}$ as $\varepsilon \to 0$ where

$$\chi = \frac{1 + c\sigma_n^2}{\cos(r_0\sigma_n/\sqrt{\eta})} = \frac{\sqrt{\eta}\sigma_n}{\sin(r_0\sigma_n/\sqrt{\eta})} = (-1)^n\sqrt{(1 + ac\lambda_n)^2 + a^2\lambda_n/b}. \quad (49)$$

Thus, in the zero-range limit we have obtained the matrix Λ in form (9) where χ and g are given by equations (49) and (48), respectively.

5. Discrete resonance sets with $pl \to$ const > 0 and $qr \to$ const > 0: case (iii)

For this case both pl and qr are supposed to tend to non-zero constants as $\varepsilon \to 0$. As above, it follows from asymptotics (22) that the first limit $pl \to$ const implies $\mu = 2$ while the second limit $qr \to$ const leads to the condition $\gamma = \nu/2$. Therefore, for this case we have to use the expansions of equations (20) with $\mu = 2$ and $\gamma = \nu/2$. As a result, one can write the following asymptotics for Λ_{21} in this case:

$$\begin{aligned}\Lambda_{21} = {} & \frac{\sigma}{l}\sinh\sigma\cos\left(\frac{\rho\sigma}{\sqrt{\eta}}\right) - \frac{\sigma}{\sqrt{\eta}}l^{-\nu/2}\cosh\sigma\sin\left(\frac{r_0\sigma}{\sqrt{\eta}}\right)\\ & - \frac{c\sigma^2}{\sqrt{\eta}}l^{\tau-1-\nu/2}\sinh\sigma\sin\left(\frac{r_0\sigma}{\sqrt{\eta}}\right)\\ & + \frac{k^2}{2}\left(\frac{1}{\sqrt{\eta}}l^{2-\nu/2} - r_0\sqrt{\eta}\,l^{\nu-1}\right)\sinh\sigma\sin\left(\frac{r_0\sigma}{\sqrt{\eta}}\right)\\ & + \frac{ck^2}{2}l^{\tau-1-\nu/2}\left\{\frac{1}{\sqrt{\eta}}\sin\left(\frac{r_0\sigma}{\sqrt{\eta}}\right)(\sinh\sigma + \sigma\cosh\sigma)\,l^2\right.\\ & \left. - \sinh\sigma\left[\sqrt{\eta}\sin\left(\frac{r_0\sigma}{\sqrt{\eta}}\right) + \rho\sigma\cos\left(\frac{r_0\sigma}{\sqrt{\eta}}\right)\right]l^\nu\right\} + \cdots. \end{aligned} \quad (50)$$

The first three terms are more singular, whereas the next terms with the factor k^2 are less singular. Therefore, we have to analyze all the possibilities when the cancellation of singularities occurs among the first three terms. Similarly, here we again have the two possibilities of cancellation of divergences in the zero-range limit.

5.1. Resonances in the $\tau \geqslant 2(\mu - 1)$ half-space: subcase (iiia)

Consider first the possibility of obtaining a finite zero-range limit of Λ_{21}, when the first two terms in (50) cancel out, while the third term vanishes or tends to a non-zero finite constant. This situation occurs if and only if $\nu = 2$ and $\tau \geqslant 2$. The cancellation happens if the equation

$$\tan\left(\frac{r_0\sigma}{\sqrt{\eta}}\right) = \sqrt{\eta}\tanh\sigma \qquad \text{or} \qquad \tan(r_0\sqrt{b\lambda}) = \sqrt{a/b}\tanh\sqrt{a\lambda} \quad (51)$$

is fulfilled. This equation admits a countable set of roots $\{\sigma_n\}_{n=0}^\infty$ or $\{\lambda_n\}_{n=0}^\infty$ where each root depends on two parameters: $\sigma_n = \sigma_n(\eta, r_0)$. Note that the first root σ_0 lying in the interval $(0, \pi/2)$ exists if and only if $r_0 < \eta$. Consequently, from asymptotics (50), after the cancellation and using equation (51), at the resonance values σ_n, $n \in \mathbb{N}$, we find

$$g = \lim_{\varepsilon\to 0}\Lambda_{21} = \frac{(-1)^{n+1}\sigma_n^2\sinh^2\sigma_n}{\sqrt{\cosh^2\sigma_n + \eta\sinh^2\sigma_n}}\begin{cases}0 & \text{for} \quad M_6\\ c & \text{for} \quad N_6\end{cases} \quad (52)$$

with the sets

$$M_6 \doteq \{\mu = \nu = 2, \tau > 2\} \qquad \text{and} \qquad N_6 \doteq \{\mu = \nu = \tau = 2\} \tag{53}$$

shown in figure 4. One can see from this expression that the constant g is a function of three parameters: $g = g(a, b, c)$. Next, it is easy to check that $\lim_{\varepsilon\to 0} \Lambda_{12} = 0$.

The $\varepsilon \to 0$ limits of the matrix elements Λ_{11} and Λ_{22} are obtained from the expansions

$$\begin{aligned}\Lambda_{11} &= (\cosh\sigma + c\sigma\varepsilon^{\tau-1}\sinh\sigma)\cos\left(\frac{r_0\sigma}{\sqrt{\eta}}\right) + \sqrt{\eta}\sinh\sigma\sin\left(\frac{r_0\sigma}{\sqrt{\eta}}\right) + \cdots,\\ \Lambda_{22} &= \cosh\sigma\cos\left(\frac{r_0\sigma}{\sqrt{\eta}}\right) - \left(\sinh\sigma + c\sigma\varepsilon^{\tau-1}\cosh\sigma\right)\frac{\sin(r_0\sigma/\sqrt{\eta})}{\sqrt{\eta}} + \cdots\end{aligned} \tag{54}$$

for any $\tau > 0$. When $\tau \geqslant 2$, the terms with the factor c vanish. Using for this case equation for resonances (51), one finds $\lim_{\varepsilon\to 0} \Lambda_{11} = \chi$ and $\lim_{\varepsilon\to 0} \Lambda_{22} = \chi^{-1}$ with

$$\chi = \frac{\cosh\sigma_n}{\cos(r_0\sigma_n/\sqrt{\eta})} = \frac{\sqrt{\eta}\sinh\sigma_n}{\sin(r_0\sigma_n/\sqrt{\eta})} = (-1)^n\sqrt{\cosh^2\sigma_n + \eta\sinh^2\sigma_n}. \tag{55}$$

Thus, the matrix Λ takes the same form as in (9) with χ and g given by equations (55) and (52), respectively.

5.2. *Resonances on the $\tau = \mu - 1$ plane: subcase (iiib)*

If we assume that only the first and the third terms in (50) cancel out, occurring at $\tau = \nu/2$, the second term becomes infinite in the $\varepsilon \to 0$ limit due to positivity of ν. The second and the third terms also cannot cancel out because they are of the same sign. Therefore, the only remaining possibility is the case when all the first three terms in (50) cancel out simultaneously. For this case to be accomplished we obtain the necessary conditions $\nu = 2$ and $\tau = 1$. We denote

$$P_6 \doteq \{\mu = \nu = 2, \tau = 1\} \tag{56}$$

and this point is shown in figure 4. The condition for resonances in this case becomes

$$\tan\left(\frac{r_0\sigma}{\sqrt{\eta}}\right) = \frac{\sqrt{\eta}\tanh\sigma}{1 + c\sigma\tanh\sigma}, \tag{57}$$

which also admits a countable set of roots $\{\sigma_n\}_{n=0}^{\infty}$ where each root depends on three parameters: $\sigma_n = \sigma_n(\eta, c, r_0)$. Note, as before, the root σ_0 exists if and only if $r_0 < \eta$.

As follows from expansion (50), at the resonances $\lim_{\varepsilon\to 0} \Lambda_{12} = 0$ and $\lim_{\varepsilon\to 0} \Lambda_{21} = 0$, i.e. in connection matrix (9) we have $g = 0$. Applying the next equation (57) to asymptotics (54) with $\tau = 1$, we obtain as above that $\lim_{\varepsilon\to 0} \Lambda_{11} = \chi$ and $\lim_{\varepsilon\to 0} \Lambda_{22} = \chi^{-1}$, but now

$$\begin{aligned}\chi &= \frac{\cosh\sigma_n + c\sigma_n\sinh\sigma_n}{\cos(r_0\sigma_n/\sqrt{\eta})} = \frac{\sqrt{\eta}\sinh\sigma_n}{\sin(r_0\sigma_n/\sqrt{\eta})}\\ &= (-1)^n\sqrt{(\cosh\sigma_n + c\sigma_n\sinh\sigma_n)^2 + \eta\sinh^2\sigma_n}.\end{aligned} \tag{58}$$

Thus, in this subcase the connection matrix Λ is also of form (9) with χ given by equation (58) and $g = 0$.

6. A resonance level with $pl \to$ const > 0 and $qr \to 0$: case (iv)

As follows from expansions (22), for this case we have the limit $pl \to \sigma$ which occurs if $\mu = 2$ while the $qr \to 0$ limit implies the inequality $\gamma > \nu/2$. Thus, instead of expansions

(24), (36) or (50), we have

$$\Lambda_{21} = \frac{\sigma}{l}\sinh\sigma\left(1 - \frac{q^2r^2}{2}\right) - \cosh\sigma\, q^2 r\left(1 - \frac{q^2r^2}{6}\right) - \sigma\sinh\sigma\, q^2 r\left(1 - \frac{q^2r^2}{6}\right) c l^{\tau-1} + \cdots. \tag{59}$$

Here again the two possibilities of cancellation of divergences take place.

6.1. *A resonance level in the $\tau \geqslant 2(\mu - 1)$ half-space: subcase (iva)*

Now we arrange the group of the most singular terms in expansion (59) as

$$\Lambda_{21}^{(0)} = \frac{\sigma}{\varepsilon}\sinh\sigma - \frac{r_0\sigma^2}{\eta}\cosh\sigma\,\varepsilon^{\gamma-\nu}. \tag{60}$$

Then for $\lim_{\varepsilon\to0}\Lambda_{21}^{(0)}$ to be finite, it is necessary to assume $\gamma = \nu - 1$ leading to the asymptotics $q^2 r = r_0\eta^{-1}\sigma^2\varepsilon^{-1} + \mathcal{O}(\varepsilon^{\nu-1})$ and the equation

$$\tanh\sigma = r_0\sigma/\eta. \tag{61}$$

This equation admits a single solution σ_0 (or λ_0) if $r_0 < \eta$. Next, from expansion (59) we pick out the less singular group

$$\Lambda_{21}^{(1)} = -\frac{\sigma}{l}\sinh\sigma\frac{q^2r^2}{2} + \frac{q^4r^3}{6}\cosh\sigma - \frac{cr_0\sigma^3}{\eta}\sinh\sigma\, l^{\tau-2} = \frac{r_0^2\sigma^3}{2\eta}\left(\frac{r_0\sigma}{3\eta}\cosh\sigma - \sinh\sigma\right)\varepsilon^{\nu-3} - \frac{cr_0\sigma^3}{\eta}\sinh\sigma\,\varepsilon^{\tau-2}. \tag{62}$$

It follows from this expression that the two inequalities $\nu \geqslant 3$ and $\tau \geqslant 2$ are necessary for $\Lambda_{21}^{(1)}$ to be finite as $\varepsilon \to 0$. According to these inequalities together with the equation $\mu = 2$, one can define the following sets shown in figure 4:

$$\begin{aligned} \Pi_0 &\doteq \{\mu = 2, \nu > 3, \tau > 2\},\\ \Pi_1 &\doteq \{\mu = 2, \nu = 3, \tau > 2\},\\ \Pi_2 &\doteq \{\mu = 2, \nu > 3, \tau = 2\},\\ \Pi_3 &\doteq \{\mu = 2, \nu = 3, \tau = 2\}. \end{aligned} \tag{63}$$

On these sets, the matrix element $g = g(\lambda_0)$ defined as a zero-range limit of $\Lambda_{21}^{(1)}$ is given by the equations

$$g = -\frac{a^2r_0^2\lambda_0^2}{\eta\sqrt{\eta^2 - ar_0^2\lambda_0}}\begin{cases} 0 & \text{for} \quad \Pi_0\\ r_0/3 & \text{for} \quad \Pi_1\\ c & \text{for} \quad \Pi_2\\ c + r_0/3 & \text{for} \quad \Pi_3. \end{cases} \tag{64}$$

Using the equation for resonances (61) together with the equality $\gamma = \nu - 1$ and inequalities $\nu \geqslant 3, \tau \geqslant 2$, one finds the zero-range limits of the other elements of the matrix Λ: $\lim_{\varepsilon\to0}\Lambda_{12} = 0$ and $\lim_{\varepsilon\to0}\Lambda_{11} = \lim_{\varepsilon\to0}\Lambda_{22}^{-1} = \chi = \chi(\lambda_0)$ where

$$\chi = \cosh\sigma_0 = \frac{\eta}{\sqrt{\eta^2 - r_0^2\sigma_0^2}}. \tag{65}$$

Thus, in the $\tau \geqslant 2(\mu-1)$ half-space or more precisely on the plane $\{\mu = 2, \nu \geqslant 3, \tau \geqslant 2\}$ the connection matrix Λ has the same form (9) with χ and g given by equations (65) and (64), respectively. The resonance set in the λ-space consists of one point being a root of equation (61).

6.2. A resonance level on the $\tau = \mu - 1$ plane: subcase (ivb)

Let us now rearrange the more singular terms in expansion (59) into the group where the ρ-term is involved. Thus, instead of expansion (60) one can write

$$\Lambda_{21}^{(0)} = \frac{\sigma}{\varepsilon}\sinh\sigma - \frac{r_0\sigma^2}{\eta}\cosh\sigma\,\varepsilon^{\gamma-\nu} - \frac{cr_0\sigma^3}{\eta}\sinh\sigma\,\varepsilon^{\gamma-\nu-1+\tau}. \tag{66}$$

Here from the equality of powers of ε we obtain the two conditions $\gamma = \nu - 1$ and $\tau = 1$. Next, all the divergences in (66) cancel out if the equation

$$\tanh\sigma = \frac{r_0\sigma}{\eta - cr_0\sigma^2} \tag{67}$$

holds. Again, if $r_0 < \eta$, this equation admits a single root σ_0. Using the equations $\gamma = \nu - 1$ and $\tau = 1$, the group of terms $\Lambda_{21}^{(1)}$ can be represented in the form

$$\begin{aligned}\Lambda_{21}^{(1)} &= -\frac{\sigma}{\varepsilon}\sinh\sigma\frac{q^2r^2}{2} + \frac{q^4r^3}{6}\cosh\sigma + c_1\sigma\sinh\sigma\frac{q^4r^3}{6}\\ &= \frac{r_0^2\sigma^3}{6\eta^2}(-3\eta\sinh\sigma + r_0\sigma\cosh\sigma + cr_0\sigma^2\sinh\sigma)\varepsilon^{\nu-3} + \cdots.\end{aligned} \tag{68}$$

Using now equation (67), expansion (68) can be transformed into

$$\Lambda_{21}^{(1)} = -\frac{r_0^3\sigma_0^4}{3\eta\sqrt{\left(\eta - cr_0\sigma_0^2\right)^2 - r_0^2\sigma_0^2}}\varepsilon^{\nu-3} + \cdots. \tag{69}$$

It follows immediately from (69) that the necessary condition for the existence of a finite limit of $\Lambda_{21}^{(1)}$ as $\varepsilon \to 0$ is the inequality $\nu \geqslant 3$.

Thus, we define the two sets

$$\begin{aligned}\Pi_4 &\doteq \{\mu = 2, \nu > 3, \tau = 1\},\\ \Pi_5 &\doteq \{\mu = 2, \nu = 3, \tau = 1\}\end{aligned} \tag{70}$$

and according to equation (69) one can write for the $\varepsilon \to 0$ limit of Λ_{21}

$$g = -\frac{r_0^3\sigma_0^4}{3\eta\sqrt{\left(\eta - cr_0\sigma_0^2\right)^2 - r_0^2\sigma_0^2}}\begin{cases}0 & \text{for} \quad \Pi_4\\ 1 & \text{for} \quad \Pi_5.\end{cases} \tag{71}$$

Concerning the other elements of the matrix Λ, using equation (67) together with the conditions $\gamma = \nu - 1$, $\tau = 1$ and the inequality $\nu \geqslant 3$, we obtain as above the $\varepsilon \to 0$ limits: $\Lambda_{12} \to 0$, $\Lambda_{11} \to \chi$ and $\Lambda_{22} \to \chi^{-1}$ with

$$\chi = \frac{\eta\cosh\sigma_0}{\eta - cr_0\sigma_0^2} = \frac{\eta}{\sqrt{\left(\eta - cr_0\sigma_0^2\right)^2 - r_0^2\sigma_0^2}}. \tag{72}$$

Thus, on the $\tau = \mu - 1$ plane or more precisely on the line $\{\mu = 2, \nu \geqslant 3, \tau = 1\}$ the connection matrix Λ has the same form (9) with χ and g given by equations (72) and (71), respectively. The resonance set in the λ-space consists of one point being a root of equation (67). Note, when $c = 0$, this equation reduces to (61).

To conclude this section, one should emphasize that in case (iv) only one resonance level $\sigma_0 = \sqrt{a\lambda_0}$ is possible if $r_0 < \eta$. In the opposite case, $r_0 \geqslant \eta$, there are no resonances and the system is completely opaque.

7. A surface of δ'-potentials

So far we have studied a wide class of point interactions with a power regularization, without any interest to which distributions they describe. The only general property used in our study above was the presence of a barrier and a well parametrized according to equations (21). The zero-range limit of any rectangular barrier–well pair $\Delta'_\varepsilon(x)$ leads to the $\delta'(x)$ potential in the particular cases when limits (3) hold. Below we shall find all the sets in the $\{\mu, \nu, \tau\}$-space where the $\delta'(x)$ potential does exist and then distinguish the corresponding δ'-potentials from all the families of point interactions studied in the previous sections.

To describe the whole class of rectangular sequences of type (18) which converge to the $\delta'(x)$ function in the sense of distributions, we calculate for any test function $\varphi(x) \in C_0^\infty$ the integral

$$\langle \Delta'_\varepsilon, \varphi\rangle = -\frac{C_\varepsilon}{(l+r)/2+\rho}[D_\varepsilon \varphi(r/2+\rho) - \varphi(-l/2) + \mathcal{O}(l^2, r^2)] \tag{73}$$

where

$$C_\varepsilon \doteq lh\left(\frac{l+r}{2}+\rho\right) \qquad \text{and} \qquad D_\varepsilon \doteq \frac{rd}{lh}. \tag{74}$$

In order to get the limit $\langle \Delta'_\varepsilon, \varphi\rangle \to -\varphi'(0)$, we have to impose in equation (73) the limits $C_\varepsilon \to C_0 = 1$ and $D_\varepsilon \to D_0 = 1$ as $\varepsilon \to 0$. In the particular case of the rectangles with parametrization (21) these limits are transformed into

$$C_\varepsilon = \frac{a}{2}(1 + r_0\varepsilon^{\nu-\mu})\varepsilon^{2-\mu} + ac\varepsilon^{1-\mu+\tau} \to 1 \qquad \text{and} \qquad D_\varepsilon = \frac{r_0}{\eta}\varepsilon^{\mu-1-\nu+\gamma} \to 1, \tag{75}$$

respectively.

The first of limits (75) can be realized on the following sets (see figure 5):

$$\begin{aligned}
&P_6 \doteq \{\mu = \nu = 2, \tau = 1\} && \text{if} \quad (1+\eta)/2 + c = 1/a,\\
&P_7 \doteq \{1 \leqslant \mu < 2, \nu = 2(\mu-1), \tau = \mu - 1\} && \text{if} \quad \eta/2 + c = 1/a,\\
&P_8 \doteq \{\mu = 2, \nu > 2, \tau = 1\} && \text{if} \quad 1/2 + c = 1/a,\\
&Q_0 \doteq \{1 \leqslant \mu < 2, \nu > 2(\mu-1), \tau = \mu - 1\} && \text{if} \quad ac = 1,\\
&Q_1 \doteq \{1 \leqslant \mu < 2, \nu = 2(\mu-1), \tau > \mu - 1\} && \text{if} \quad a\eta = 2,\\
&Q_2 \doteq \{\mu = 2, \nu > 2, \tau > 1\} && \text{if} \quad a = 2,\\
&Q_3 \doteq \{\mu = \nu = 2, \tau > 1\} && \text{if} \quad 1+\eta = 2/a.
\end{aligned} \tag{76}$$

All these sets form a closed trihedral surface $S_{\delta'} \doteq P_6 \cup P_7 \cup P_8 \cup Q_0 \cup Q_1 \cup Q_2 \cup Q_3$ in the $\{\mu, \nu, \tau\}$-space being an analog of the line $L_{\delta'}$ in the $\{\mu, \nu\}$-space. Consider now which sets considered in the previous sections belong to $S_{\delta'}$. One finds that $M_4 \cup M_5 \cup N_4 \cup N_5 \subset Q_1$ (iia), $\cup_{j=0}^{j=3}\Pi_j \subset Q_2$ (iva), $M_6 \cup N_6 \subset Q_3$ (iiia) and $\cup_{j=0}^{j=3}P_j \subset Q_0$ (ib), $P_4 \cup P_5 \subset P_7$ (iib). However, the constraints in the right column of (76) make narrow cases (ii) and (iii).

The second of limits (75) is fulfilled if $\gamma = 1 - \mu + \nu$ and $r_0 = \eta$. The first of these equalities holds for all the cases considered above because in cases (ii) and (iii) $\gamma = \nu/2$ and $\nu = 2(\mu - 1)$ and in case (iv) $\gamma = \nu - 1$ and $\mu = 2$. The second equality is valid in case (ia) while in cases (ii) and (iii) r_0 is arbitrary. In case (ib) we have

$$D_0 = \frac{r_0}{\eta} = \frac{1}{1+\lambda} < 1 (ac = 1) \tag{77}$$

and r_0 must be less η in case (iv) in order to provide the roots of equations (61) or (67). Therefore $D_0 = 1$ in all the cases except for (ib) and (iv).

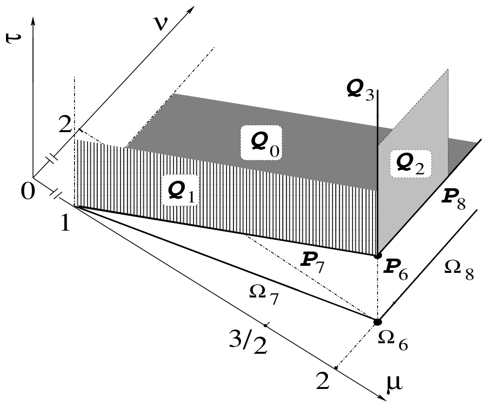

Figure 5. Trihedral surface $S_{\delta'}$ obtained from the $\varepsilon \to 0$ distributional limit of regularizing sequence (18) with parametrization (21). This surface is formed by apex P_6, edges P_7, P_8, Q_3 and planes Q_0, Q_1, Q_2. For two dimensions ($\tau \equiv 0$) point Ω_6, lines Ω_7 and Ω_8 form the closed line $L_{\delta'}$ where the $\delta'(x)$ function is obtained from the same regularizing sequence (18) but now with $\rho \equiv 0$.

Thus, the $\delta'(x)$ function is well defined only if $C_0 = 1$ and $D_0 = 1$ and these equalities can be satisfied for cases (ii) and (iii). Due to inequalities (27), we have $C_0 = 0$ for case (ia) describing the δ-interaction and $D_0 < 1$ for cases (ib) and (iv). Note that limits (3) hold if $(C_0, D_0) = (1, 1)$. However, using equation (77), one can re-define parametrization (21) for a non-zero and non-resonant δ'-potential in case (ib) as follows:

$$l = \varepsilon, \quad h = a\varepsilon^{-\mu}, \quad r = \frac{\eta}{1+\lambda}\varepsilon^{1-\mu+\nu}, \quad d = (1+\lambda)b\varepsilon^{-\nu}, \quad \rho = a^{-1}\varepsilon^{\mu-1} \tag{78}$$

involving here the coupling constant λ. Then $D_0 = 1$, limits (3) take place and the corresponding sequence $\Delta'_\varepsilon(x)$ converges to the $\delta'(x)$ function in the sense of distributions.

8. Concluding remarks

Thus, using representation (21) of the rectangular barrier–well parameters l, h, d, r and ρ via the powers μ, ν and τ (the fourth power γ is expressed in terms of these three powers) of the squeezing parameter ε, we have constructed in the $\varepsilon \to 0$ limit a whole family of point interactions which includes δ- and δ'-potentials as particular cases. The third parameter τ determines how rapidly the barrier and the well are approaching each other as $\varepsilon \to 0$. One can distinguish three sets in the $\{\mu, \nu, \tau\}$-space with $1 < \mu \leqslant 2$ and $\nu > 0$: the planes $\tau \equiv 0$, $\tau = \mu - 1$ and the half-space $\tau \geqslant 2(\mu - 1)$. The plane $\tau \equiv 0$ corresponds to the case when the barrier–well distance is zero ($\rho = 0$ or $c = 0$). This case studied earlier in [14] corresponds to the second repeated limit (5). The $\tau = \mu - 1$ plane is isolated from the $\tau \geqslant 2(\mu - 1)$ half-space. Particularly, the point $\{\mu = \nu = 2, \tau = 1\}$ which corresponds to $\rho = c\varepsilon$ is illustrated in figure 1 by path 3 while the line $\{\mu = \nu = 2, \tau \geqslant 2\}$ corresponds to all the paths running below path 4 including this one for which $\rho = c\varepsilon^2$. Note that the case with the $\tau \geqslant 2(\mu - 1)$ half-space cannot be fitted in regularization (7). Nevertheless, the convergence proof as $\varepsilon \to 0$ seems to be possibly extended to the case of an appropriately deformable function $v(\varepsilon; \xi)$ during the limiting procedure.

The construction of point interactions has been realized in this paper via approximating singular potential (2) by regular potential (18) using the transfer matrix approach to yield the explicit solutions in the form of matrix elements (20). This approach is useful when applied for the Hamiltonian

$$\mathcal{H}_\varepsilon = -\mathrm{d}^2/\mathrm{d}x^2 + \lambda\Delta'_\varepsilon(x) \tag{79}$$

with $\Delta'_\varepsilon(x)$ being a compactly supported piecewise constant function. For a general case of $\Delta'_\varepsilon(x)$ defined according to (7), Hamiltonian (79) can be proved to converge in the norm resolvent sense to the limiting Hamiltonian $\mathcal{H} = \mathcal{H}(\lambda)$ with the domain $\{\psi(x) \in W_2^2(\mathbb{R}\backslash\{0\}) | \psi(+0) = \chi\psi(-0), \psi'(+0) - \chi^{-1}\psi(-0) = g\psi(-0)\}$ where λ belongs to a resonant set and $W_2^2(\mathbb{R}\backslash\{0\})$ stands as usual for the Sobolev space of functions which belong to $L_2(\mathbb{R}\backslash\{0\})$ together with their derivatives up to the second order. In this way the norm resolvent convergence proof has been realized for Hamiltonian (79) used as a model for approximating bent quantum waveguides by quantum graphs under regularization (7) with the assumption $\int_{\mathbb{R}} v(\xi)\,\mathrm{d}\xi \neq 0$ [8, 9]. The result in [8] coincides with that obtained in section 5.2 where $\chi \neq 1$ and $g = 0$. It has been extended in [9] to include a more general limit with $g \neq 0$. Very recently [18] $\mathcal{H}_\varepsilon$ has been proved to converge to $\mathcal{H}$ as $\varepsilon \to 0$ in the norm resolvent sense under the δ'-like properties: $\int_{\mathbb{R}} v(\xi)\,\mathrm{d}\xi = 0$ and $\int_{\mathbb{R}} \xi v(\xi)\,\mathrm{d}\xi = -1$.

The calculations which realize the $\varepsilon \to 0$ limit of the Schrödinger equation with a rectangular-like potential using the transfer matrix approach, for instance, Λ with elements (20), seem to be more simple compared to the analysis of the norm resolvent convergence of Hamiltonian (79). In our case it is sufficient to carry out properly the cancellation of divergences emerging from the kinetic energy operator $-\mathrm{d}^2/\mathrm{d}x^2$ and the potential $\lambda\Delta'_\varepsilon(x)$ in the $\varepsilon \to 0$ limit only for the (most singular) element Λ_{21}. Due to this cancellation, $\mathcal{H}$ cannot be represented as the sum of the kinetic energy operator $-\mathrm{d}^2/\mathrm{d}x^2$ and the singular potential $\lambda\delta'(x)$. The sum of these operators is well defined only if $\varepsilon > 0$ when the domain of $\mathcal{H}_\varepsilon$ is the space $W_2^2(\mathbb{R})$, the functions of which are continuous together with their first derivatives at the origin $x = 0$. However, a function $\psi(x)$ continuous at $x = 0$ except for $\psi(x) \equiv 0$ cannot fulfill the limiting equation $\mathcal{H}\psi = E\psi$ because the product $\delta'(x)\psi(x)$ in the sense of distributions is $\psi(0)\delta'(x) - \psi'(0)\delta(x)$. Therefore, the domain of the total Hamiltonian $\mathcal{H}$ cannot coincide with the domain of the kinetic energy operator $-\mathrm{d}^2/\mathrm{d}x^2$ alone defined on the whole axis $\mathbb{R}$. Concerning the potential term $\delta'(x)\psi(x)$, it is ambiguous for any wavefunction $\psi(x)$ discontinuous at $x = 0$. This ambiguity means that in the Schrödinger equation with the δ'-potential there exists a hidden (in general, multi-dimensional) parameter fixing a regularizing sequence $\Delta'_\varepsilon(x)$.

As demonstrated throughout the paper, the cancellation proceeds in different ways in the $\tau \geqslant 2(\mu - 1)$ half-space (a-subcases) and on the $\tau = \mu - 1$ plane (b-subcases). As a result, the limiting transfer matrix Λ which connects the two-sided boundary conditions for the wavefunction $\psi(x)$ at $x = \pm 0$ takes the form as in equation (9). In fact, the cancellation of singularities occurs even in the volume M_0 (or on the plane Ω_0 in the case of two dimensions) where Λ equals the matrix unit I (full transmission). The surface boundary S_δ of this volume (or the line boundary L_δ of Ω_0) splits the regions of full and zero transmission allowing on this boundary the existence of the δ-interaction with the effective coupling constant g given by equations (29). Next, in the region of zero transmission the 'islands' with resonant tunneling (see sections 4 and 5) and a partial non-zero and non-resonant transmission (see section 3.2) appear on the trihedral surface $S_{\delta'}$ describing all the ways of regularizing the $\delta'(x)$ function under assumption (21). For all types of a partial transmission, in equation (9) we have $\chi \neq 1$.

Finally, it should be stressed that for all the ways of three-dimensional parametrization (21), $\mu \in (1, 2]$, resulting in the existence of point interactions with a non-zero (complete or partial, resonant or non-resonant) transmission, there exists the cancellation of divergences described in detail in each section. However, the resulting family of point interactions is not the only case where the cancellation of divergences in the zero-range limit takes place. Thus, the similar cancellation procedure has been realized to construct δ-like point interactions in two [20] and three [21] dimensions as exactly solvable models. These models describe a quantum particle moving in the two- or three-dimensional space from which the origin is removed. They are not so rich as one-dimensional point interactions because the allowed self-adjoint extensions in two and three dimensions depend only on one parameter, instead of four parameters in one dimension, as given by connection matrix (12). However, the key point in the regularization procedure for the point interactions in one, two and three dimensions is the same: (i) the existence of non-trivial point interactions is a result of the cancellation of divergences coming from the kinetic and potential energy terms and (ii) the domain of the limiting total Hamiltonian are the functions which belong neither to the domain of the kinetic energy operator nor to the domain of the potential energy operator. A similar situation is known not only in the theory of differential operators with singular pertubations but also in the quantum field theory under constructing (non-trivial and renormalized) quantized Hamiltonians in different models such as the Yukawa$_2$ and ϕ_3^4 interactions (for more details see [22] as well as the comments with extensive bibliography on the constructive quantum field theory given in [23]).

Acknowledgments

We thank Yu D Golovaty for sending us his manuscripts prior to publication, careful reading of this manuscript and bringing to our attention paper [9]. His valuable suggestions, comments and critical remarks are greatly acknowledged. We are grateful to P L Christiansen, Y B Gaididei and S V Iermakova for stimulating and helpful discussions. Partial support from the Ukrainian State grant for Fundamental Research is also acknowledged.

References

[1] Demkov Yu N and Ostrovskii V N 1975 *Zero-Range Potentials and Their Applications in Atomic Physics* (Leningrad: Leningrad University Press)
Demkov Yu N and Ostrovskii V N 1988 *Zero-Range Potentials and Their Applications in Atomic Physics* (New York: Plenum)
[2] Albeverio S, Gesztesy F, Høegh-Krohn R and Holden H 2005 *Solvable Models in Quantum Mechanics (With an Appendix Written by Pavel Exner)* 2nd revised edn (Providence, RI: AMS Chelsea Publishing)
[3] Albeverio S and Kurasov P 2000 *Singular Perturbations of Differential Operators: Solvable Schrödinger-Type Operators* (Cambridge: Cambridge University Press)
[4] Avron J E, Exner P and Last Y 1994 *Phys. Rev. Lett.* **72** 896
[5] Exner P 1995 *Phys. Rev. Lett.* **74** 3503
[6] Exner P and Šeba P 1996 *Phys. Lett.* A **222** 1
[7] Cheon T, Exner P and Šeba P 2000 *Phys. Lett.* A **277** 1
[8] Albeverio S, Cacciapuoti C and Finco D 2007 *J. Math. Phys.* **48** 032103
[9] Cacciapuoti C and Exner P 2007 *J. Phys. A: Math. Theor.* **40** F511
[10] Šeba P 1986 *Rep. Math. Phys.* **24** 111
[11] Patil S H 1994 *Phys. Scr.* **49** 645
[12] Coutinho F A B, Nogami Y and Perez J F 1997 *J. Phys. A: Math. Gen.* **30** 3937
[13] Christiansen P L, Arnbak H C, Zolotaryuk A V, Ermakov V N and Gaididei Y B 2003 *J. Phys. A: Math. Gen.* **36** 7589
[14] Zolotaryuk A V, Christiansen P L and Iermakova S V 2006 *J. Phys. A: Math. Gen.* **39** 9329
[15] Toyama F M and Nogami Y 2007 *J. Phys. A: Math. Theor.* **40** F685
[16] Zolotaryuk A V 2008 *Adv. Sci. Lett.* **1** 187
[17] Golovaty Yu D and Man'ko S S 2009 *Ukr. Math. Bull.* (in Ukrainian) **6** 173 (arXiv:0909.1034v [math.SP])

[18] Golovaty Yu D and Hryniv R O 2009 arXiv:0911.1046v [math.SP]
[19] Albeverio S, Dąbrowski L and Kurasov P 1998 *Lett. Math. Phys.* **45** 33
[20] Perez J F and Coutinho F A B 1991 *Am. J. Phys.* **59** 52
[21] Coutinho F A B and Amaku M 2009 *Eur. J. Phys.* **30** 1015
[22] Glimm J and Jaffe A 1971 *Some Quantum Field Models: Statistical Mechanics and Quantum Field Theory* ed C deWitt and R Stora (New York: Gordon and Breach)
[23] Reed M and Simon B 1975 *Methods of Modern Mathematical Physics* vol 2 (New York: Academic)